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CALCULATION OF THE SPATIAL  
DIFFUSION OF SMALL PARTICLES IN  
AN EXPONENTIAL ATMOSPHERE

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CALCULATION OF THE SPATIAL DIFFUSION  
OF SMALL PARTICLES IN AN EXPONENTIAL ATMOSPHERE

by Milton M. Klein, Kwang Yu, and Richard H. Harrison

1. Introduction

The problem of general diffusive motion of small particles through an atmosphere of variable density has been treated recently by several investigators. A numerical solution for the case of an exponential atmosphere has been made by Banister and Davis<sup>1</sup>. An analytic solution motivated by their results, has been obtained by Granzow<sup>2</sup> in the form of an infinite series of Laguerre polynomials. In one analysis of the  $F_2$  layer of the ionosphere as a charged system in which diffusion occurs in the ambipolar limit, Yonezawa<sup>3</sup> has obtained a solution which may be applied to neutral particles whose mass is half that of the ambient particles. These analyses have been limited to the case of one-dimensional motion. Recently Klein and Yu<sup>4</sup> have obtained analytic solutions for both three-dimensional and one-dimensional motion in atmospheres having exponential and parabolic density distributions. The numerical work was confined to the one-dimensional problem for the exponential atmosphere. An extension of the calculation to the three-dimensional problem is presented here for the exponential atmosphere.

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## 2. Analysis

The diffusion of small particles in an atmosphere whose density varied exponentially with altitude  $z$  was studied in detail in Reference

4. The diffusion coefficient  $D$  was taken in the form

$$\frac{D}{D_0} = e^{z/H} \quad (1)$$

where  $D_0$  is a reference value of  $D$  at  $z = 0$ , and  $H$  is the scale height.

An analytic solution for the three-dimensional density distribution

$n(\vec{r}, t)$  was obtained in the form<sup>4</sup>

$$\frac{n(\vec{r}', t')}{n_0} = \frac{1}{8\pi H^3 t'} \zeta_0^{1-a} \zeta^{1+a} \left[ \exp - \left( \frac{\zeta^2 + \zeta_0^2}{t'} \right) \right] \int_0^\infty I_\beta \left( \frac{2\zeta\zeta_0}{t'} \right) J_0 \left( \frac{\beta \rho}{2} \right) \rho d\rho \quad (2)$$

where  $n_0$  is the total number of particles  $\zeta = e^{-z/2H}$ ,  $\zeta_0 = e^{-z_0/2H}$ ,  $\sigma^2 = (x' - x'_0)^2 + (y' - y'_0)^2$ ,  $x'_0, y'_0, z'_0$ , is the point of release,  $\beta^2 = \sigma^2 + (a-1)^2$ , and  $a$  is the ratio of the mass of a foreign particle to that of an ambient particle. In Equation (2) the unit of length is  $H$  while the time  $t'$  is measured in units of  $H^2/D_0$  (dimensionless values of  $\vec{r}$  and  $t$  are primed).

Numerical calculation of the density  $n$  requires the evaluation of the definite integral

$$N = \int_0^\infty I_\beta(\lambda) J_0 \left( \frac{\beta \rho}{2} \right) \rho d\rho, \quad (3)$$

$$\lambda = \frac{2\zeta\zeta_0}{t}$$



occurring in Equation (2). Because of the unusual requirement of integration of a Bessel function with respect to a variable which depends upon the order of the Bessel function, an exact determination of the integral  $N$  does not appear possible. An approximate analytic solution for the late times may, however, be obtained. For large values of  $t$  the parameter  $\lambda \ll 1$  and  $I_\beta(\lambda)$  may be expanded in a power series

$$I_\beta(\lambda) = \frac{\left(\frac{\lambda}{2}\right)^\beta}{\Gamma(\beta+1)} + \frac{\left(\frac{\lambda}{2}\right)^{\beta+2}}{\Gamma(\beta+2)} + \dots \quad (4)$$

and, upon retention of the first two terms, Equation (3) becomes,

$$N = \int_0^\infty \frac{e^{-A\rho}}{\Gamma(\beta+2)} \left( \beta + 1 + \left(\frac{\lambda}{2}\right)^2 \right) J_0\left(\frac{B\rho}{2}\right) \rho d\rho \quad (5)$$

The solution of Equation (3) therefore depends upon expressing the gamma function  $\Gamma(\beta+2)$  in a suitable form. The fact that for large  $x$  the asymptotic form of the gamma function is

$$\Gamma(x) \sim x^{-\frac{1}{2}} \left( x \log x \right)^{-x} \quad (6)$$

suggests that a suitable form for  $\Gamma(\beta+2)$  that yields a tractable integral which is reasonably accurate is

$$\Gamma(\beta+2) \approx c \beta e^{b\beta} \quad (7)$$

where  $c$  and  $b$  are constants which can be evaluated by matching Equation (7) to the exact gamma function at two suitable points. Equation (7) is most accurate at large  $\beta$ , i.e., for large  $a$ , but tends to give too large values near  $a = 1$ . Utilization of the integral

$$\int_0^{\infty} \frac{e^{-b\beta}}{\beta} J_0\left(\frac{\sigma}{2}\beta\right) \rho d\rho = \frac{e^{-\alpha \sqrt{b^2 + \sigma^2/4}}}{\sqrt{b^2 + \sigma^2/4}} \quad (8)$$

where  $\alpha = a-1$  yields for the integral N

$$N = \frac{1}{c} \left[ \left( 1 + \frac{\lambda^2}{4} \right) I_1 + I_2 \right] \quad (9)$$

where

$$I_1 = \frac{e^{-\alpha \sqrt{R}}}{R} \quad (10a)$$

$$I_2 = \frac{(A+b)e^{-\alpha \sqrt{R}}}{R} \left[ \alpha + \frac{1}{R} \right] \quad (10b)$$

$$R^2 = (A+b)^2 + \sigma^2/4 \quad (10c)$$

The integral N therefore exhibits an exponential behaviour  $e^{-\frac{a-1}{2}\sigma}$  for large  $\sigma$ .

For the case  $a = 1$ , where  $\beta = \sigma$ , the exponential vanishes and the integral may be represented as follows:

$$N = \left[ 1 + \frac{\lambda^2}{4} \right] I_1 + (1+c) I_2 + c I_3 \quad (11)$$

$$I_1 = \frac{\lambda + b}{R^{3/2}} \quad (12a)$$

$$I_2 = \frac{2[(A+b)^2 - 3\sigma^2/4]}{R^{5/2}} \quad (12b)$$

$$I_3 = \frac{3(A+b)[2(A+b)^2 - 3\sigma^2/4]}{R^{7/2}} \quad (12c)$$

### 3. Results and Discussion

Some typical results obtained from this investigation are presented in Figures 1 through 9. In Figures 1 through 3 we have plotted the relative density  $n/n_{\max}$  ( $n_{\max} = n$  at  $\sigma = 0$ ) as a function of  $\sigma$  for several typical values of  $a$  at 600, 400, and 150 km. In all cases we note, as anticipated, that the heavier particles exhibit a more rapid drop-off in density, regardless of the altitude, than do the lighter particles. It should not be inferred, however, that the absolute density at a particular location for a heavy particle is always below that for a light one.

Absolute density contours in the  $z - \sigma$  plane for  $a = 1, 2, 5$  and  $10$  are represented in Figures 4 through 9 at time  $t' = 10^{-4}$ . We note that as the altitude becomes lower the densities for the heavier particles ( $a = 5$ , for instance) are greater than those for the light particles ( $a = 1$ , for example). A comparison of the  $a = 1$  and  $a = 5$  contours (Figures 3 and 7) indicates the very rapid spreading out of the lighter particles and the extremely small dispersion for the heavier particles. These figures indicate that the lighter the particles the greater is their horizontal spread, as one would expect. In the case  $a = 5$ , we have also included results for  $t' = 10^{-2}$  and  $t' = 10^{-6}$ .

Taking  $D_0 \sim 400 \text{ cm}^2/\text{sec}$  and  $H \sim 20 \text{ km}$ ,  $t' = 10^{-4}$  corresponds to  $t = 10^6 \text{ sec} \simeq 12 \text{ days}$ . Assuming  $n_0 = 10^{26}$  particles and  $z_0 = 400 \text{ km}$ , we have at  $t' = 10^{-4}$  the following table:

a	Approximate altitude of peak density at $\sigma = 0$	$H\sigma$ (in km)				
		200	400	600	1000	2000
1	190 km	$4 \times 10^{17}/\text{cc}$	$10^{17}/\text{cc}$	$3 \times 10^{16}/\text{cc}$	$2 \times 10^{15}/\text{cc}$	$7 \times 10^{13}/\text{cc}$
2	170 km	$5 \times 10^{17}$	$5 \times 10^{15}$	$3 \times 10^{13}$	$7 \times 10^8$	less than 1/cc
5	150 km	$5 \times 10^{16}$	$10^8$	1/cc	- - -	- - -
10	140 km	$\sim 10^{15}$	$\sim 10^4$	less than 1/cc	- - -	- - -

The values obtained from the approximations described in Section 2 have been checked at several points by direct numerical integration of the integral N using an IBM 1620 computer for  $a = 1, 2, 5$  and 10. The agreement of the two has been found to be satisfactory.

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- 1) Banister, J. and H. Davis, Physics of Fluids 5, 136 (1962).
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- 3) Yonezawa, T., J. Radio Research Labs (Japan), 1 No. 3 and No. 4 (1954); 2, 125 (1955).
- 4) Klein, M.M. and K. Yu, Calculation of the Diffusion of Small Particles in a Non-uniform Atmosphere, GCA Technical Report 62-1-G, June, 1962.

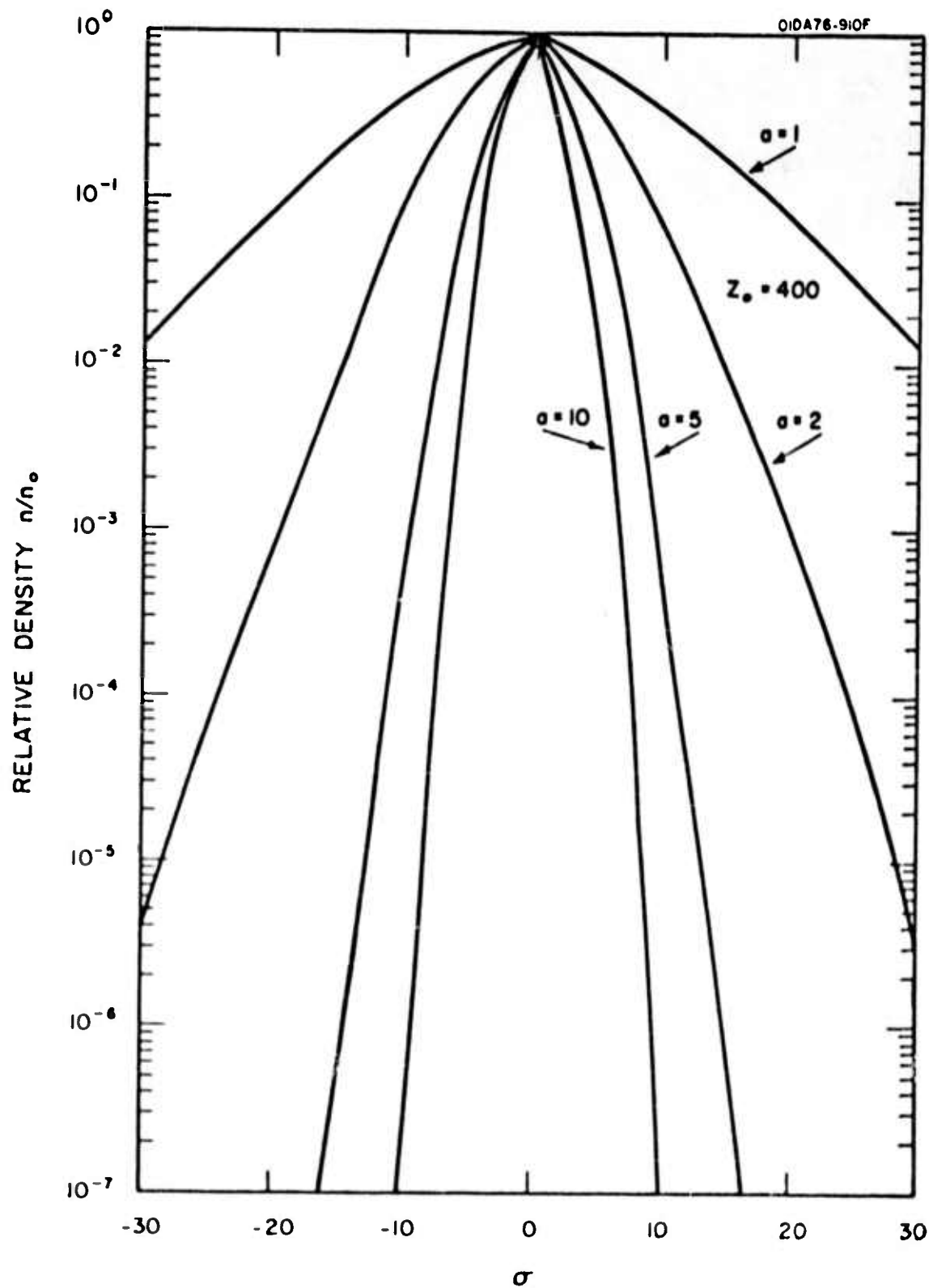


Figure 1. Relative density  $\frac{n}{n_{\max}}$  for several values of  $a$ ;  
 $Z = 150$  km,  $H = 20$  km,  $\frac{D_0}{H^2} t = 10^{-4}$ .

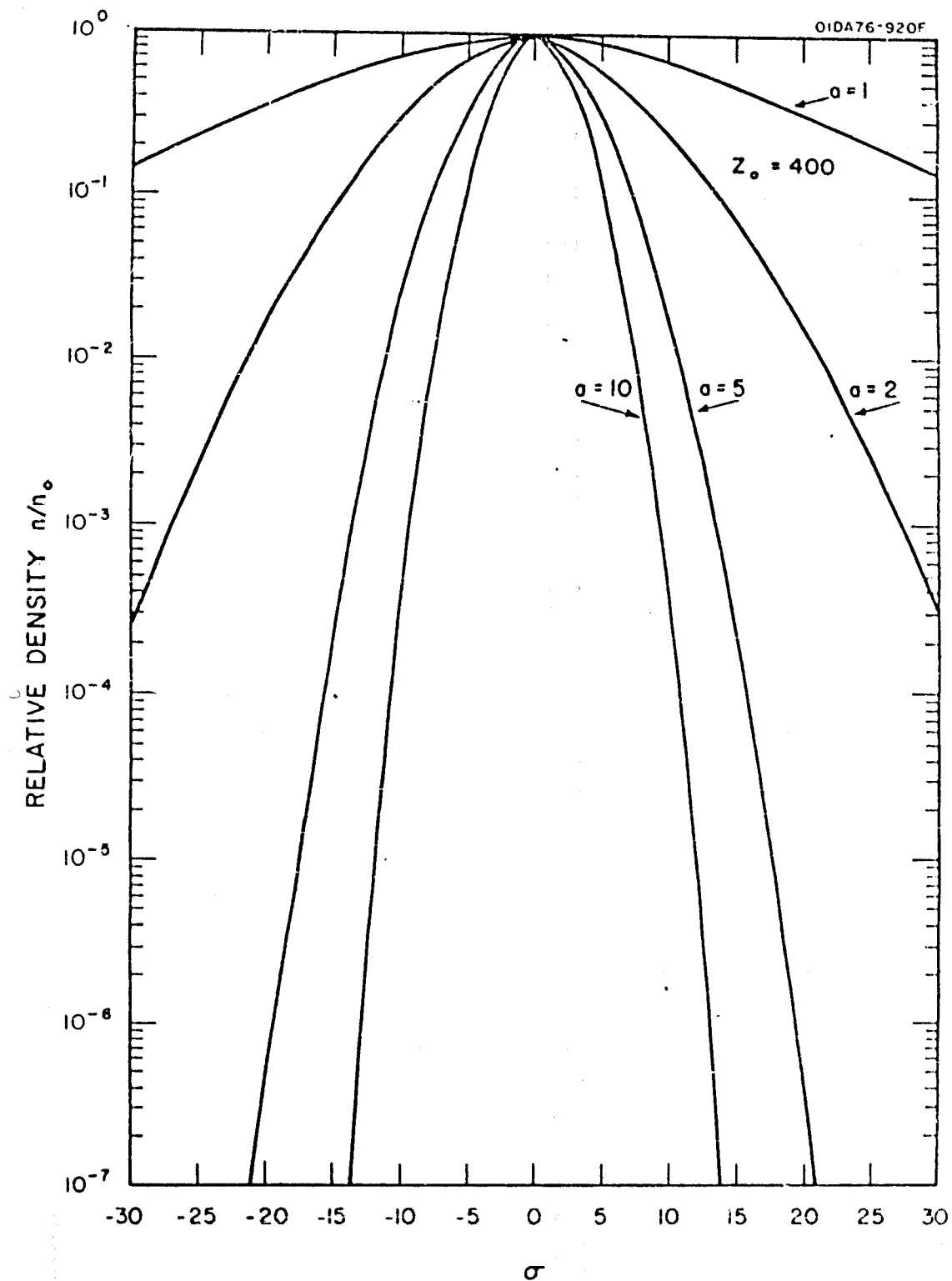


Figure 2. Relative density  $\frac{n}{n_{\max}}$  for several values of  $\alpha$ ;  
 $Z = 400$  km,  $H = 20$  km,  $\frac{D_0}{H^2} \tau = 10^{-4}$ .

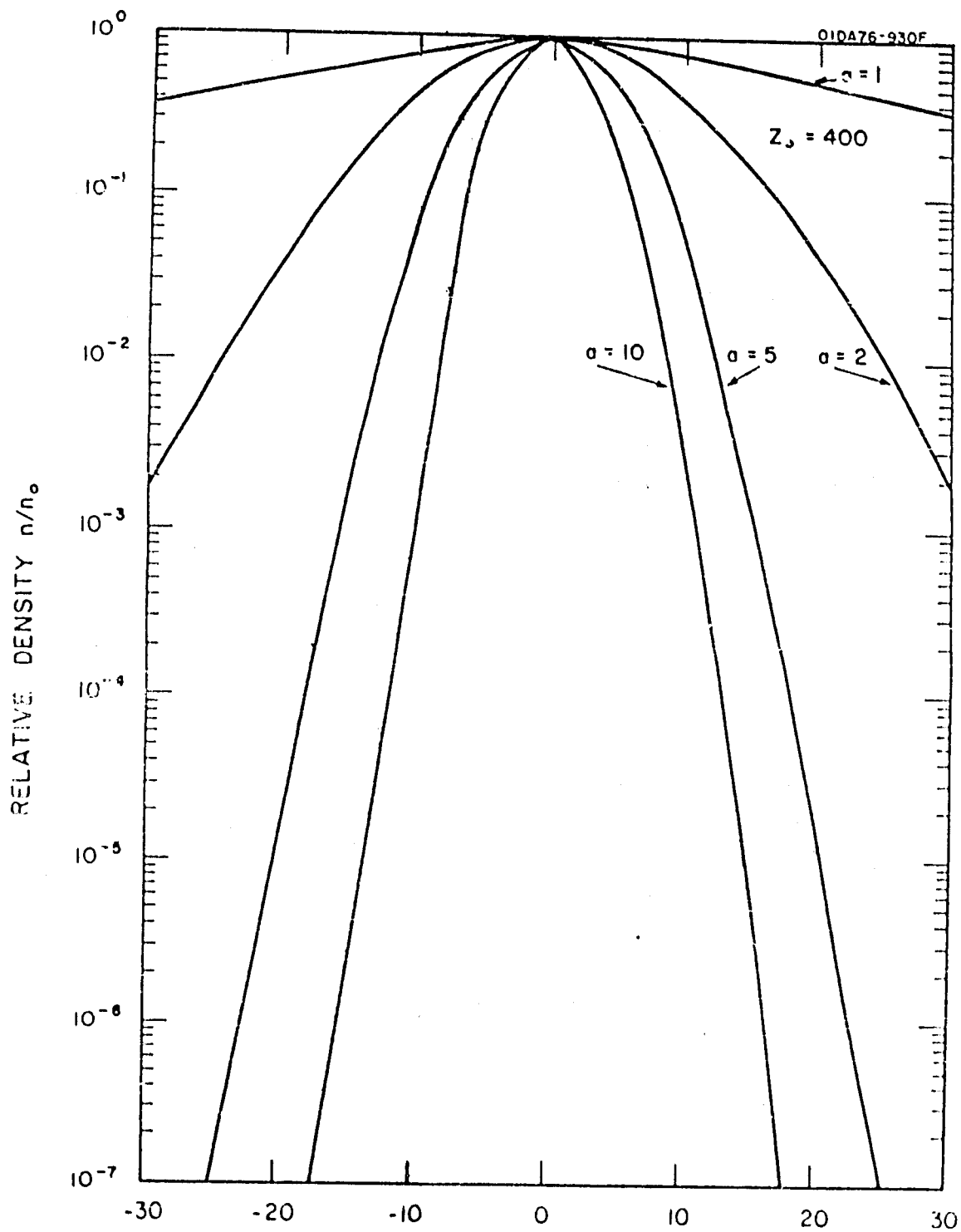


Figure 3. Relative density  $\frac{n}{n_{\max}}$  for several values of  $\alpha$ ;  
 $Z = 600$  km,  $H = 20$  km,  $\frac{D_0}{H^2} t = 10^{-4}$ .



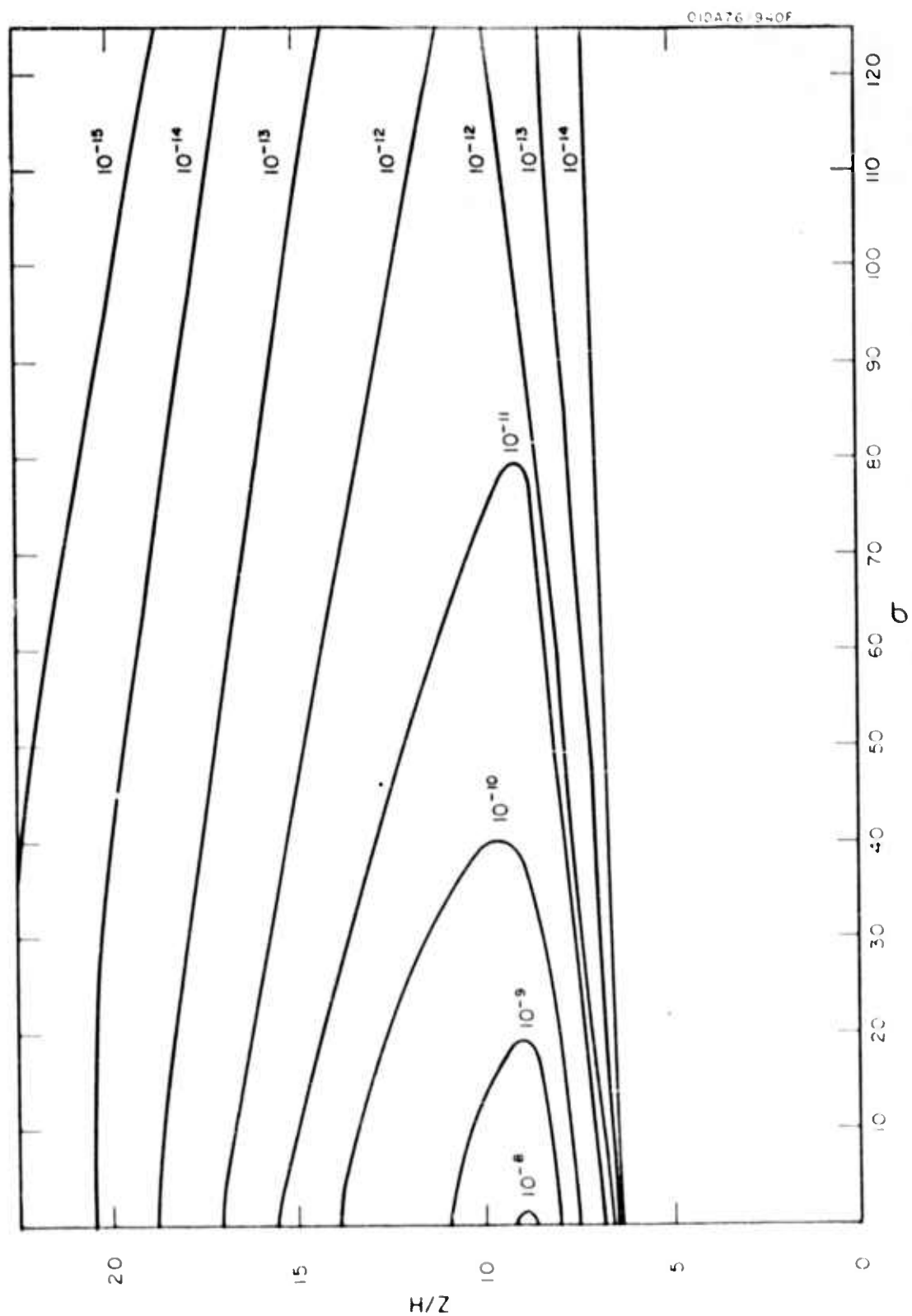


Figure 4. Absolute density contours. Mass ratio  $a = 1$ ;  
 $H = 20 \text{ km}$ ,  $\frac{\rho}{\rho_0} t = 10^{-4}$ .

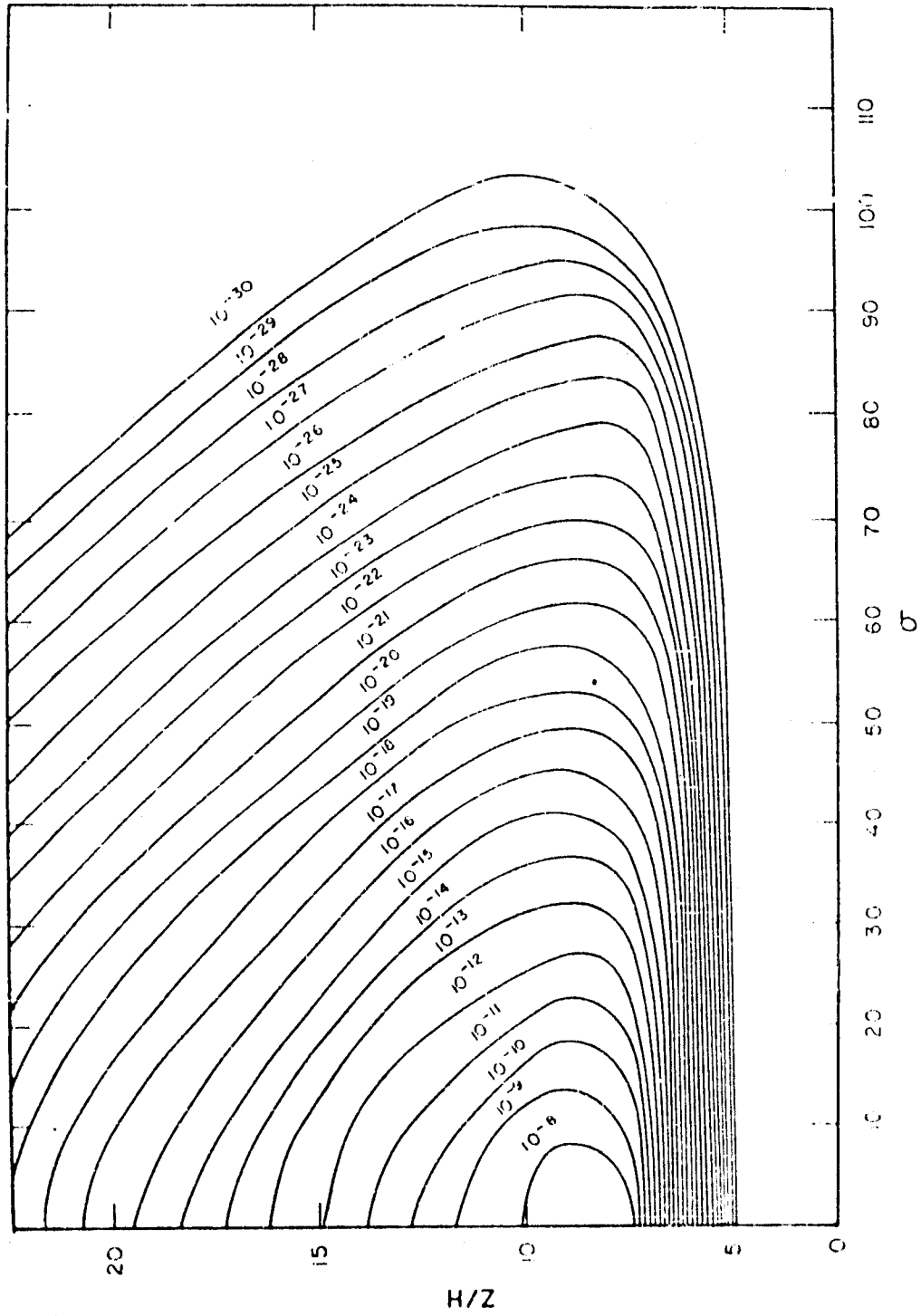


Figure 5. Absolute density contours. Mass ratio  $a = 2$ ;  
 $\frac{2}{3} \frac{Z}{A} = 10^{-4}$ .  
 $\mu = 20$  km;  $\frac{2}{3} \frac{Z}{A} = 10^{-4}$ .

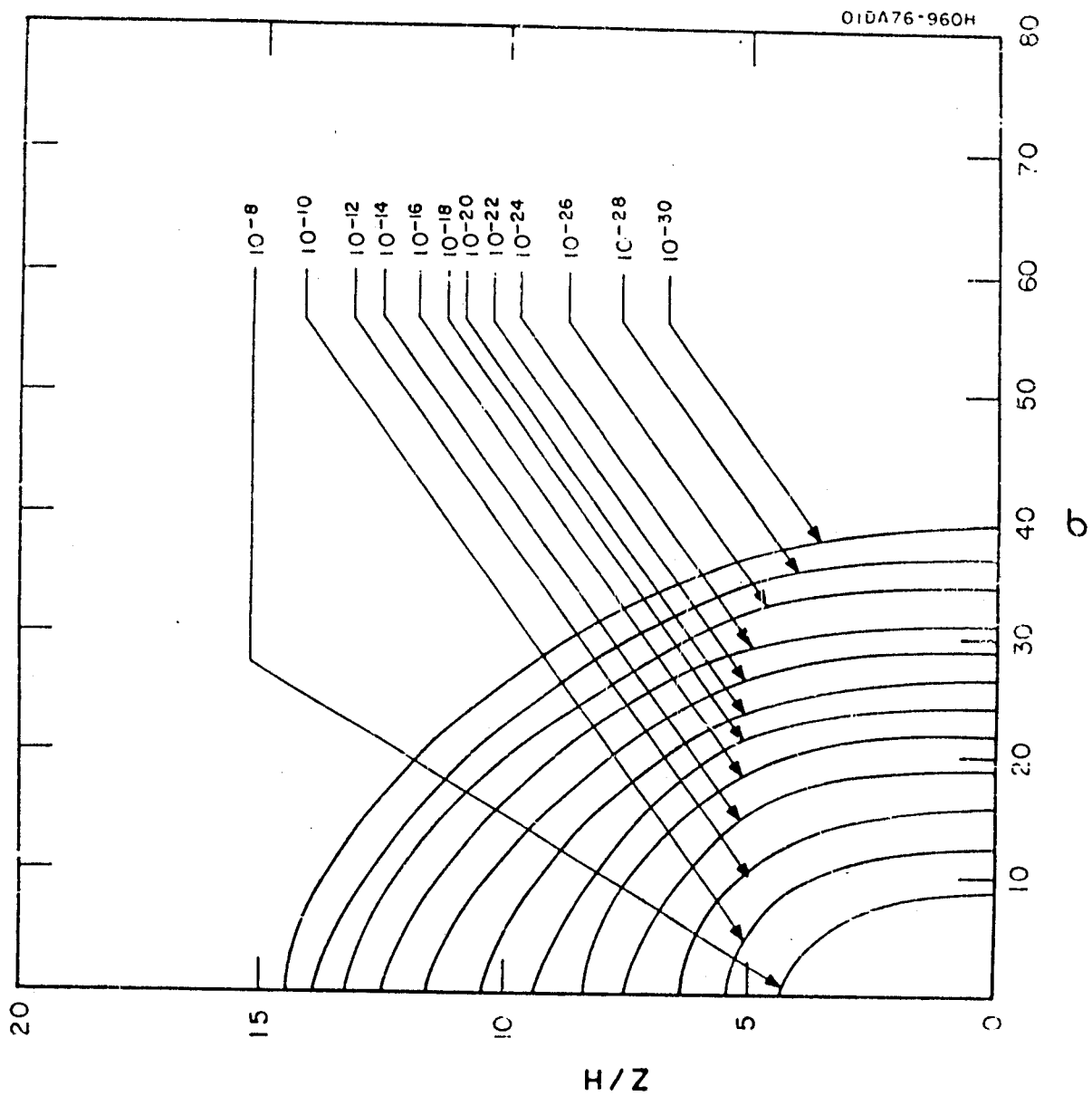


Figure 6. Absolute density contours. Mass ratio  $a = 5$ ;  
 $H = 20$  km;  $\frac{1}{H^2} \tau = 10^{-2}$ .

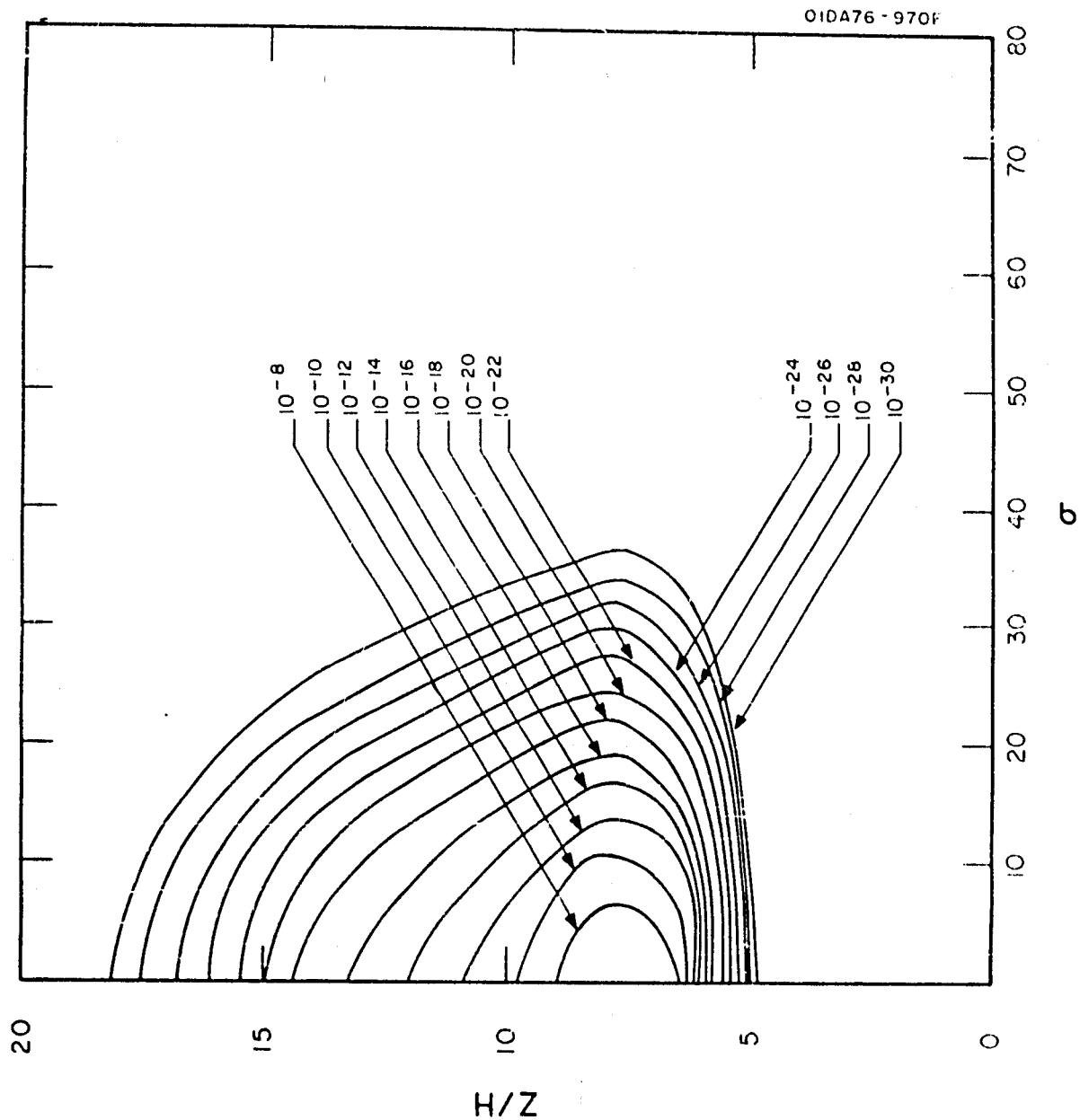


Figure 7. Absolute density contours. Mass ratio  $a = 5$ ;  
 $H = 20$  km;  $\frac{D_0}{H^2} t = 10^{-4}$ .

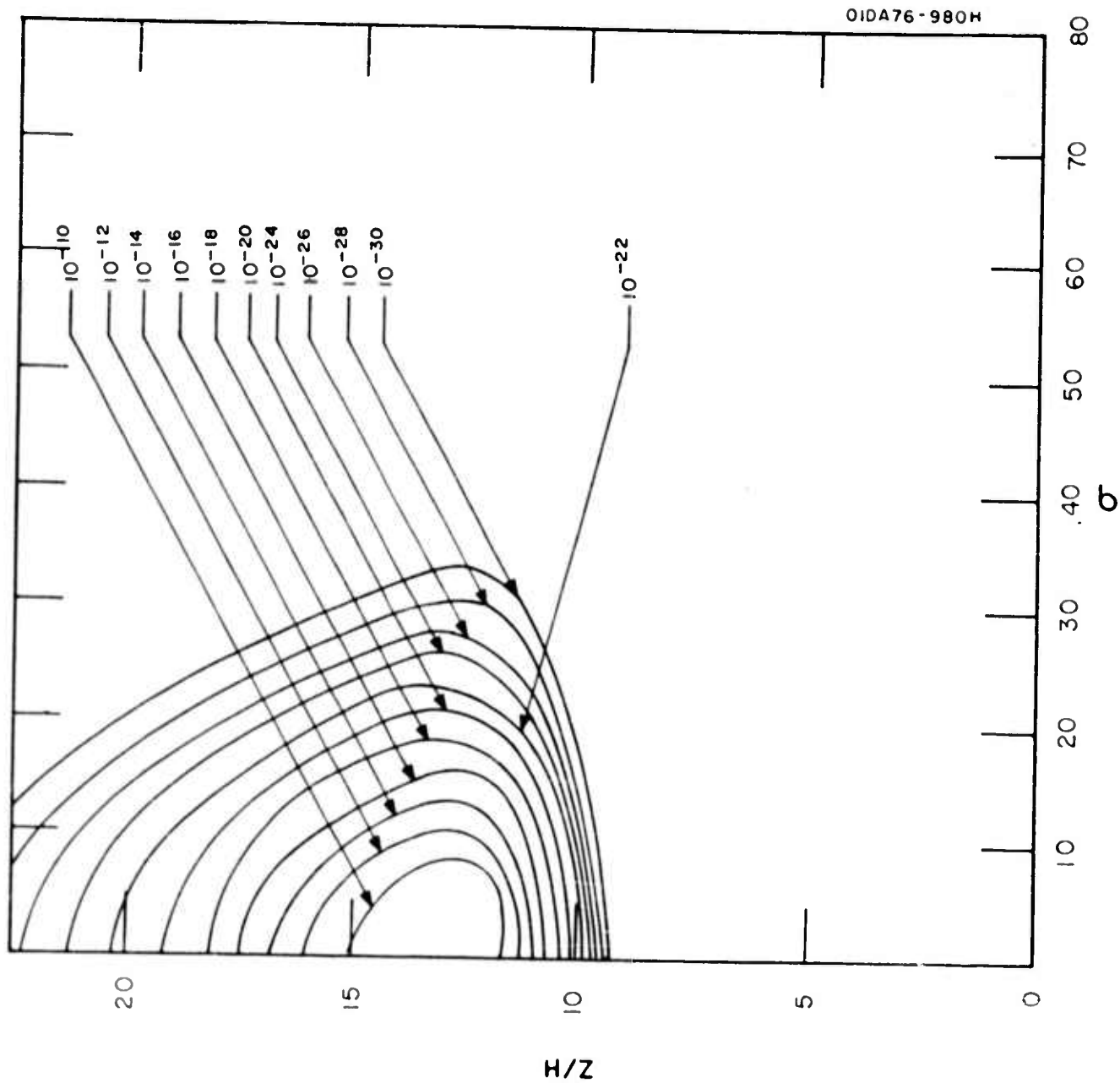


Figure 8. Absolute density contours. Mass ratio  $a = 5$ ;  
 $H = 20$  km;  $\frac{D_0}{H^2} t = 10^{-6}$ .

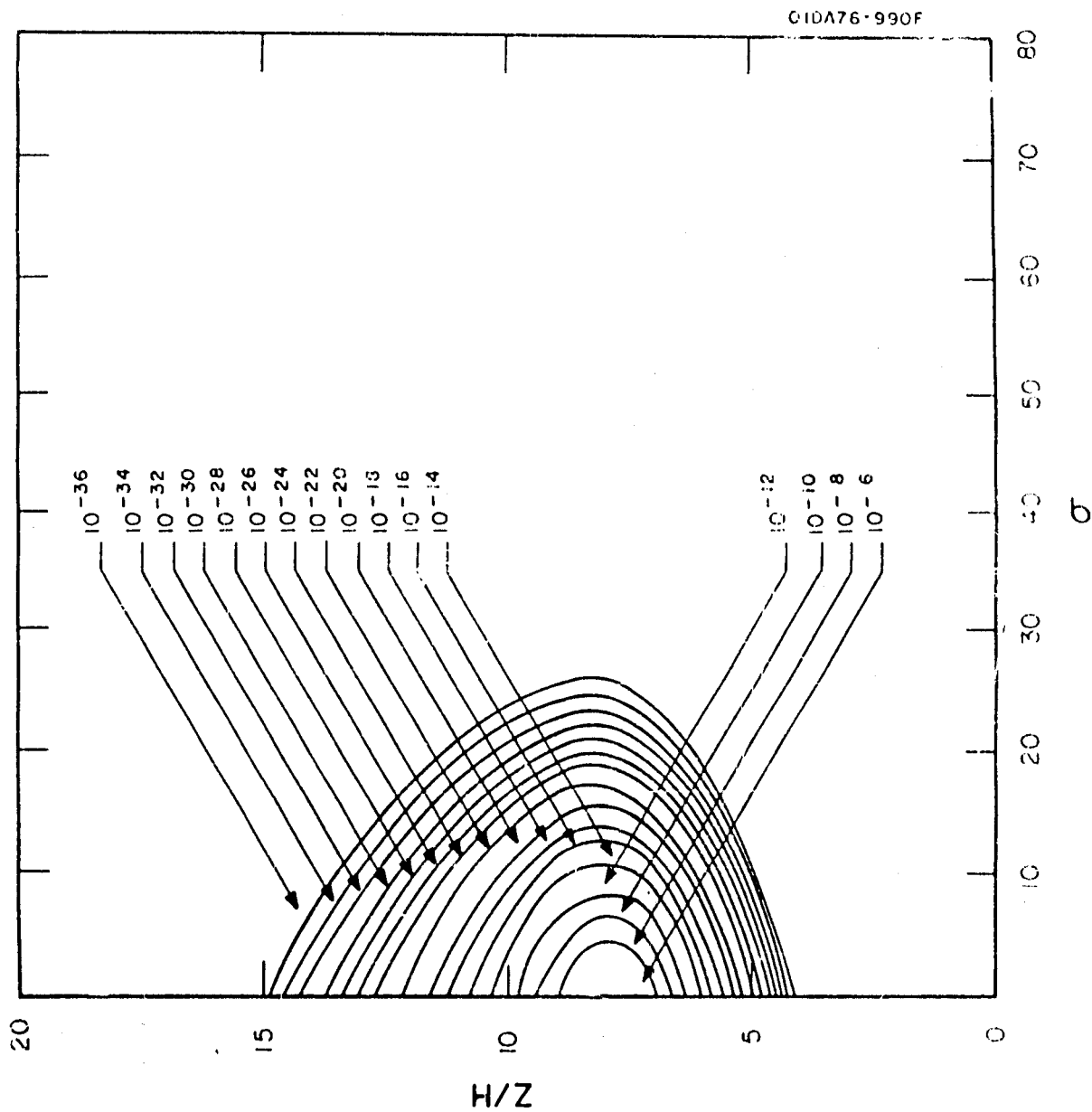


Figure 9. Absolute density contours. Miss ratio  $\mu = 10$ .  
 $H = 20$  km;  $\frac{p}{H^2} t = 10^{-4}$ .

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